Investigating Student Use of Representation in Quantum Mechanics Change of Basis Problems

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Meta-Representational Competence (MRC), first theorized by Andrea diSessa, has been used widely in Math Education studies. It has been recently used in Physics Education Research in the domain of Quantum Mechanics. Specific problems within Quantum Mechanics have a foundation in Linear Algebra, and may be approached or perceived differently based on the notation used (either Dirac, Matrix, or Spinor notation). Semi-structured interviews were conducted to present physics students with content questions and then ask them about MRC concepts directly, as it related to the content questions or other situations in Physics or Math. Student statements were coded to indicate both previously identified and novel facets of MRC in the context of change of basis problems in Quantum Mechanics. The preliminary analysis of two student interviews demonstrates each student's distinct utility of MRC concepts, and suggests extending the Wawro, Watson, & Christensen (2020) MRC statement codes.

Keywords: Quantum Mechanics, Meta-Representational Competence, Physics, Linear Algebra, Representation.

Literature Review

Notions involving representation and notation in Quantum Mechanics have often been investigated in Physics Education Research (PER). Fredlund, Airey & Linder (2012) noted that "the affordances of different representations determine the role they can play in communication, and thus in the sharing of knowledge." These affordances, or the "inherent potential of that representation to provide access to disciplinary knowledge", are said to "enable certain representations to become legitimate within a discipline such as physics. Physics learning then, involves coming to appreciate the disciplinary affordances of representations." (Fredlund, Airey & Linder, 2012). Additionally, Gire & Price (2015) identified the terms "*individuation, degree of externalization, compactness, and symbolic support*" as "structural features of quantum notations". This structural features framework was adapted by Schermerhorn et al. (2019) into a "Computational Features Framework", in order to "answer questions regarding how and why students use different methods". While PER literature has examined representations in Quantum Mechanics in the past decade, it often did so without leveraging diSessa's Meta-Representational Competence (MRC) Theoretical Framework (2002), (2004).

Investigations using MRC have been recently conducted in the context of Quantum Mechanics, specifically in units involving "spin" which tend to use Eigentheory concepts. In a 2020 study, Wawro et al. enumerated the different MRC concept-related codes that arose in their interviews which were structured around Eigentheory concepts. These codes were in part informed by the "Computational Features Framework" developed by Schermerhorn et al (2019), along with additional codes that were organized and explained through the MRC framework.

The Wawro et al. (2020) study has been cited in a recent paper by Corsiglia et al. (2022), which focused on QM Change of Basis problems. However, the Corsiglia study focused on a phenomenographical approach and did not implement the MRC theoretical framework used in the Wawro study. This current research seeks to incorporate MRC at a boundary between realms of RUME and PER in the specific context of QM Change of Basis problems. This is done through the coding of MRC concept statements that involve codes listed in Wawro et al. (2020)

and novel codes that were not found in this earlier study. We define "MRC concept/statement" here as a concept/statement that could be analyzed by the MRC Theoretical Framework.

Theoretical Framework

The work presented is attempting to make direct measures of Andrea diSessa's theoretical framework, Meta-Representational Competence (diSessa, 2002). Meta-Representational Competence (MRC) "...[describes] the full range of capabilities that students (and others) have concerning the constructions and use of external representations" (diSessa & Sherin, 2000). Iszák Iszák, Çağlayan & Olive (2009) mention regarding a diSessa paper (2002), "[diSessa] observed that students' criteria usually emerge in reaction to particular examples." Iszák also noted, "he reported that different students can make systematically different judgments about external representations."

This framework has seen significant use in the Research in Undergraduate Math Education community (Hillel, 2000; Arcavi, 2003; Iszák, 2003; Iszák, Çağlayan & Olive, 2009) but has seen little use in Physics Education Research. Notions involving representation and notation in Quantum Mechanics have been often investigated in Physics Education Research (PER), but only one paper incorporated Meta-Representational Competence as a framework, Wawro et al. (2020). In Wawro et al. (2020), MRC statement codes were created based on the data observed (see Fig. 1) and proved to be very fruitful in parsing the data observed. However, the theoretical framework was applied post-hoc. This work aims to use an interview protocol designed specifically to parse out student statements regarding MRC in the context of Quantum Spin states.

A. Value-based Preference	B. Problem-based Preference
1. Clarity 2. Speed 3. Familiarity 4. "Likeability" 5. Ease of writing	1. Useful in calculations 2. Makes more direct use of given relationships 3. Needs less information 4. Compactness 5. Individuation 6. Externalization
C. Purpose and Utility Awareness	
1. Has freedom to choose symbols 2. Has an ease with notation, writes symbols to mean what is personally desired 3. Aware of one's own progress in notation use 4. Able to "step back" and weigh options to decide which notation system is best	

Figure 1. Table of MRC categories and codes from Wawro et al. (2020)

Background

In Quantum Mechanics, spin is a property of some particles, often described as "when a particle behaves like it has angular momentum (is 'spinning'), even though the particle is not physically spinning". This often confusing concept is better understood mathematically through the use of Eigentheory. Spin states of a particle or system are represented by Kets (|Psi>), which function similarly to vectors. For Spin-½ particles, the most common type in Intro-Quantum Mechanics situations, applying a directional Spin Operator can only result in one of the two possible eigenvalues and its corresponding eigenvector. The result is either "spin-up" (a "+" eigenvector with a corresponding eigenvalue of $+\frac{\hbar}{2}$) or "spin-down" (with a "-" eigenvector with eigenvalue of $-\frac{\hbar}{2}$). In a certain coordinate direction (namely z, x, or y), a spin-½ particle's spin state can be described as a linear combination of these $|+>$ "spin-up" and "spin-down"$|->$ kets.

These state vector kets, defined as linear combinations of eigenvector kets, can also be written in terms of 2 by 1 matrices. Here, the first entry represents the $|+>$ ket (usually in the z-direction), and the second entry represents the $|-\!>$ ket. Introductory units of instruction may present the same information in both Dirac and Matrix form, so that students can see how one notation is translated to the other. Additionally, "Spinor" notation is sometimes used to write statements without explicitly using a Dirac ket or a column vector for the eigenvectors.

In any notation, the relations of the eigenvector kets, the spin operators, and the eigenvalue kets can be written in eigenequation form: $\hat{S}_z\,|+\!> = \frac{\hbar}{2}|+\!>$. Figure 2 (below) includes an example of the 3 notations used in the context of a state vector equation:

Spinor: $\chi = \frac{3}{5} \cdot \chi_+ - \frac{4}{5} \cdot \chi_-$ Dirac: $|\psi\rangle = \frac{3}{5} \cdot |+\rangle - \frac{4}{5} \cdot |-\rangle$ Matrix: $\chi = \frac{3}{5} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{4}{5} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Figure 2: Equivalent state vectors, or wavefunction kets, written in different notations

Methods

A semi-structured interview was designed to provide participants with content questions and ask them about MRC concepts directly (Browner et al., 1988). Content questions were adapted from questions from the Corsiglia et al. (2022) study, which focused on Change of Basis, including concept-focused and computational-focused questions. The questions were adapted to provide a context for discussing MRC statements, rather than pure tests of conceptual understanding. For some of these content questions, multiple versions were made with different representations in the questions (See Fig. 2). After each set of content questions, a verbal follow-up question was asked. The follow-up questions were intended to target MRC concepts directly. At first, these follow-ups treaded lightly on MRC concepts and focused more on content, but as the interview progressed, the follow-ups were directly aimed at MRC concepts.

This interview structure was selected to organically ease students into talking about MRC concepts without directly explaining the MRC theoretical framework. To avoid dead-ends due to content issues, hints were structured into the content questions, or ready to be provided by the interviewer. The structure of the interview questions was meant to guide students through the most efficient path (Corsiglia et al., 2022) through answering the questions.

Three student interviews were conducted at a midwestern, land-grant research university, each lasting about one hour long. Of these 3 interviews, we elected to focus the preliminary analysis on the interviews of participants PS and CG, as their interviews were the most unlike each other. After the interviews were transcribed, one of the authors reviewed the transcriptions to code mentions of MRC statements, guided by the codes delineated in Wawro et al. (2020).

Preliminary Results

The depth and amount of MRC statements made by student participants with relatively little direct prompting were surprising and encouraging. While physics students may unknowingly make MRC-related statements with their peers (e.g., preferences for notation, etc.) while working on group assignments, there is little evidence that they are instructed or directly graded on MRC concepts within physics courses. However, the interview participants all had some form of vocabulary for expressing some MRC concepts, to varying degrees, without being explicitly introduced to the terminology. Additionally, interview participants seemed to accept and use vocabulary that was explicitly introduced during the interview with good facility.

We focused our preliminary study on Student Participants PS and CG, as they had distinct interview experiences from an MRC concept standpoint. They performed differently on the tougher content questions. They also articulated different preferences in notation. We share some preliminary analyses of each interview separately to best highlight the most prominent features of each participant's interview experience. Future work will aim to synthesize these and other interviews we've performed.

PS Interview: PS is a Physics Undergraduate student, who had taken two semesters of undergraduate Quantum Mechanics from a midwestern land-grant research university. PS identifies as a white, cis-gendered man. PS made fewer MRC comments in total, and it took them longer to 'warm up' to speaking freely about MRC in depth. PS was effective in elaborating on their thought process while working on content questions when prompted. Additionally, most of the MRC statements PS made were "Value-based preferences" (Fig. 1).

Within the first 10 minutes of the interview, PS made statements involving their belief in a "Hierarchy" of notations. PS stated that Dirac (and maybe Spinor as well) notation is just a "shorthand" for the Matrix notation. There was no existing code from Wawro et al (2020), so we defined a new code to encapsulate this. Much later in the interview, PS made a similar remark regarding the Spinor notation in Question 5 ("Q5"), after being given Question 7 ("Q7"), (emphasis added in **bold**):

PS: There's a reason that I already like this notation way better, I got to be honest.
Interviewer: Why do you like it better?
PS: But again, you know, **it's like, hey, spin up -- 1 0, spin down -- 0 1**, and like, like I said before, I just **I like the given** - what exactly **given values for just like shorthand** [points at Q5] like Psi, Psi plus that's the spin-up, and you know, I kinda get, like all screwed up in my mind, like - how do I do this again?
PS: **Where I can actually, like… work out the numbers?** Like, [points at Q5 page] can one over square root of three be multiplied times Chi spin up? **Whereas here [Q7] it's like, I know I can, you know?**

It's possible that PS views this hierarchy of Matrix notation as being the 'real' notation, with Dirac and Spinor notation as 'shorthands' because they prefer doing change of basis computations in Matrix notation. Additionally, PS made many MRC statements about "Clarity", where Matrix notation was often 'more clear', and Dirac notation was 'less clear'. They also mentioned "Likeability" often: liking Matrix notation, and disliking Dirac notation.

PS spent significant time on Q5 but was not able to complete it. They expressed that they did not have enough equations to solve for the constants they wanted. When given Q7, which was a Matrix notation focused instead of Spinor notation focused, PS responded with the reaction in the interview transcript above. Even though the problems were similar, PS was able to complete Q7. Afterwards, PS made conflicting statements regarding their success in Q7 compared to Q5. Initially, they made statements indicating that they understood the question better after they saw it presented in Matrix notation (Q7). However, a couple minutes later, they said that they couldn't be sure that the different notation made it easier. In their opinion, their success may have been due to just seeing the same question a second time. A goal of future work is to investigate the connection between struggling to answer problems and a person's understanding of concepts that fall under Meta-representational Competence.

CG Interview: CG, a current Physics graduate student, has taken Quantum Mechanics at both an Undergraduate level at a small, private college in the midwest, and at the Graduate level at a midwestern land-grant research university. CG identifies as a white, cis-gendered man. CG, who performed extremely well on content questions, spoke fluently about how they represented their work verbally, on the page through equations, and using hand gestures. Relative to PS, CG made

MRC statements earlier and more often over the course of their interview. Many of these statements involved a preference for Dirac notation. “Value-based preferences” MRC statement codes (Wawro et al. 2020) such as “Speed”, “Familarity”, “Likeability”, and “Ease of Writing” were often made by CG in support of Dirac notation and against Matrix notation. CG also made “Problem-based Preference” statements, noting Dirac notation was “useful in calculations” because it had “tons of tricks.”

While PS made few statements regarding “Purpose and Utility Awareness” (See Fig. 1), CG made many (at least 10) distinct statements in this category; the first one coming during an early follow-up question. Most frequent were statements that would fall under “Aware of one’s own progress in notation use”, which encompassed Dirac, Matrix, and Spinor notation. CG also made statements consistent with the “Able to ‘step back, and weigh options to decide which notation system is best” code after one of the verbal follow-up questions and while considering how to begin working on Q5.

CG made many statements involving concepts of “Abstractability” and “Visualizability.” To CG, Matrix notation was very visible. They used hand gestures to set up a 3-dimensional coordinate system to communicate state vectors. CG then mentioned understanding Matrix notation as applying a linear transform to a vector, to change the direction of the state vector. In contrast, CG viewed Dirac notation as being more “abstract” in nature. They stated that the Dirac Kets could represent anything, or be written as “smiley faces” or “frowny faces” without any loss of meaning. Because of CG’s focus on these statements, and that they emphasized the wording, we believe “Abstractability” and “Visualizability” may be identified as new codes for extending the Wawro, Watson & Christensen (2020) MRC codes related to expectation values (See Fig. 1).

Discussion / Conclusion

MRC concepts, which involve the capability to use external representations (diSessa & Sherin 2000), are likely related to skills developed by individuals by the time they are considered an ‘expert’ in a particular type of problem. Interviewee CG, who performed extremely well on content questions, successfully represented their work verbally, on the page through equations, and using hand gestures. Interviewee PS struggled with several content questions and did not demonstrate many MRC statements outside of the Value-based Preference category. How and when MRC concepts are developed and the role it plays alongside content understanding in Upper-division Physics is still unclear. Future research may shed additional light onto the relationship between students’ understanding of their thinking of how to use different representations and their performance on content questions in Upper-division Physics.

Questions for Audience

If students are not directly taught or coached regarding MRC concepts, how do they develop it in a Quantum Mechanics spins context which draws heavily on Eigentheory concepts?

Would directly instructing students on understanding their thinking of how to use different representations help students become adept at this and content understanding?

At what points throughout a mathematics or physics curriculum could instruction be initiated and subsequently revisited to support MRC concept development across a curriculum?

Acknowledgments

I would like to thank Dr. Megan Wawro for their expert input on this preliminary research. I am grateful to Sam Gisi, Jessica Searl, and Kristen Rodenhausen for their advice and insight throughout the research process. I would also like to thank the innumerable other individuals whose support culminated in my opportunity to conduct this research.

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